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Brownian Local Time and Quantum Mechanics

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Let V be any (sufficiently regular) attractive potential in one and two dimensions. We make rigorous an argument of M. Kac [1], relating the recurrence of the Brownian motion to the existence of at least one bound state for the quantum Hamiltonian $H = -(\Delta/2) + V$.

KEY WORDS: Brownian motion; recurrence; Local Time; bound state.

1. INTRODUCTION

There are (at least) three physical properties in spaces of low dimensions (1 or 2) which cease to hold in higher dimensions (3 or more).

First of all, a symmetric random walk (or a Brownian motion) is recurrent in one and two dimensions but is transient for dimension $d \geqslant 3$. That is, if $\{x(t), t \geqslant 0;\ x(0) = x \in \mathbb{R}^d\}$ is the d-dimensional Brownian motion (BM), then

$d \leqslant 2$ $\quad |\{t \geqslant 0: x(t)$ belongs to an open set $S\}| = +\infty$, for every S the
$d > 2$ $\quad$ same quantity is finite for every bounded S

where $|\cdot|$ is the Lebesgue measure. See, for example, reference [1, p. 80] The same property holds for a symmetric random walk on $\mathbb{Z}^d$.

The second property is: for every attractive potential V of arbitrary small depth and support, the corresponding quantum Hamiltonian H has always at least one bound state in $d = 1$ or 2. In higher dimensions, the discrete spectrum of H is empty if the parameters (depth and support) of V are small enough (Ref. 1, p. 114).

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Finally, there is no breakdown of a continuous symmetry in statistical mechanics (or in QFT) if $d=1$ or 2,[2] while this breakdown occurs if $d \geq 3$).[5]

The dimensional behavior of these three phenomena is too characteristic not to expect some relation between them.

A connection between the recurrence of BM and the existence of a bound state for any attractive potential in $d=1, 2$ was suggested by M. Kac (Ref. 1, p. 115). The main goal of this note is to complete Kac's argument into a proof of the existence of such a bound state. Our proof is not based on any explicit solution of the Schrödinger equation, but only on recurrence properties of the BM. This is the content of Section 2.

There are many links between statistical mechanics and random walks. In particular, the average number of visits to the origin of a symmetric d-dimensional random walk enters in the proof of the presence ($d \geq 3$) and of the absence ($d=1, 2$) of continuous symmetry breakdown. However, the relation between both phenomena does not seem to go much beyond this formal level.

In Section 3, we interpret our proof as an entropy–energy argument and compare it to the analogous statistical-mechanical situation.

2. EXISTENCE OF A BOUND STATE IN ONE AND TWO DIMENSIONS

2.1. Preliminaries

We first give some definitions and some results which will be used later.

Let $\{x_w(t),\ t \geq 0;\ x_w(0)=x \in \mathbb{R}^d\}$ be the d-dimensional Brownian motion ($d-$BM) and Ω_x be the space of $\mathbb{R}^d$-valued continuous $x_w(t)$, $t \geq 0$, with $x_w(0)=x$: $\Omega_x = \{w \mid x_w(t)\colon \mathbb{R}_+ \to \mathbb{R}^d,\ x_w(0)=x\}$. We define for $a \in \mathbb{R}^d$, $w \in \Omega_x$ and $t>0$ the Brownian local time $\tau(t, a, w)$ as

$$\tau(t, a, w) = \int_0^t \delta(x_w^1(s) - a^1) \dots \delta(x_w^d(s) - a^d)\, ds \tag{2.1}$$

in which $\delta(\cdot)$ is the Dirac distribution and the upper index denotes the component. Each component of a d-BM is a 1-BM.

$\tau(t, a, w)$ measures the presence of the path w at a during time t. We also define the integral Brownian local time $\varepsilon(t, w)$ as

$$\varepsilon(t, w) = \int_{\mathbb{R}^d} \tau(t, x, w)\, e(dx) \tag{2.2}$$

for any nonnegative and finite measure e on $\mathbb{R}^d$.

For $d=2$, we will be interested in a spherically symmetric potential, and so it will be useful to consider the radial component of the d-BM, namely the d-dimensional Bessel process $\{r_w(t)=|\vec{x}_w(t)|,\ t\geqslant 0; r_w(0)=l\in\mathbb{R}_+\}$. The analogues of (2.1) and (2.2) for a Bessel process read:

$$\tau(t, m, w)=\int_0^t \delta[r_w(s)-m]\, ds \tag{2.3}$$

$$\varepsilon(t, w)=\int_{\mathbb{R}_+} \tau(t, r, w)\, \mu(dr) \tag{2.4}$$

for any nonnegative and finite measure μ on $\mathbb{R}_+$.

Let us also recall the Feynman–Kac formula, for a general continuous Markov process $(S, \Omega_x, P_x; x\in S)$ characterized by its state space S, its sample path space Ω_x, and the probability measures P_x of the process starting at x.

Let θ be the generator of the process, k a positive and piecewise continuous function, and f a bounded and continuous function. Then the Feynman–Kac formula states that (Ref. 6, p. 54)

$$v(x)=E_x\int_0^\infty e^{-\alpha t}f[x_w(t)]\, e^{-\int_0^t k[x_w(s)]ds}\, dt \tag{2.5}$$

is the bounded and continuous solution of

$$[\alpha+k(x)-\theta]\, v(x)=f(x) \tag{2.6}$$

$E_x(\cdot)$ is expectation with respect to P_x.

In case of d-BM ($S=\mathbb{R}^d$, $\Omega_x=$ space defined above, $P_x=$ Wiener measure), the operator θ_{BM} is $\Delta/2$, and so the homogeneous solution of (2.6) is the eigenfunction with energy $-\alpha$ of the Hamiltonian $H=-\Delta/2+k(x)$.

Equations (2.5) and (2.6) provide an explicit form for the kernel of $\exp(-tH)$ with $H=H_0+V$ (Ref. 1, p. 53)

$$e^{-tH}(x, y)=\int_{\Omega_x} e^{-\int_0^t V[x_w(s)]\, ds}\, Db(x, 0; y, t) \tag{2.7}$$

where $Db(x, 0; y, t)$ is the unnormalized Wiener measure conditioned on the fact that motion starts at x and arrives after a time t at y.

In the case of the d-dimensional Bessel process: $S=\mathbb{R}_+$, $\Omega_l=\{w\,|\,r_w(t): \mathbb{R}_+\to\mathbb{R}_+, r_w(0)=l\}$ and $P_l=$ Bessel measure. (See [6, p. 59] for more information about Bessel processes.)

The generator of d-dimensional Bessel process is $\theta_{\mathrm{BP}} = \frac{1}{2}[d_r^2 + (d-1)/r)\, d_r] =$ the radial part of θ_{BM}.

Consider a spherically symmetric potential V in $d \geqslant 2$ and denote by H_r the radial part of H: $H_r = -\theta_{\mathrm{BP}} + V(r)$. The Feynman–Kac formula gives the analogue of (2.7) for the Bessel process.

$$e^{-tH_r}(l, m) = \int_{\Omega_l} e^{-\int_0^t V[r_w(s)]\,ds}\, Dv(l, 0; m, t) \tag{2.8}$$

in which $Dv(l, 0; m, t)$ is the unnormalized Bessel measure conditionned on the fact that the motion starts at l and arrives at m after a time t. $(t > 0; l \in \mathbb{R}_+; m \in \mathbb{R}_+ \setminus \{0\})$.

2.2. The One-Dimensional Case

We want to prove the existence of at least one bound state for any potential of the form $V = V_1 + V_2$, with V_1 in $L^2(\mathbb{R}$ or $\mathbb{R}^2)$, V_2 in $L^s(\mathbb{R}$ or $\mathbb{R}^2)$ and V_2 goes to zero as $|x|$ goes to infinity. Furthermore, V is negative and strictly negative on some open set.

It is clearly enough to prove the result for a square-well potential $V = -\lambda I\{|\vec{x}| \leqslant L\}$, for any L and λ strictly positive. We shall prove it using properties of BM and without solving the corresponding Schrödinger equation.

Let us recall some "poetry" of Kac (Ref. 1, p. 114). Let $H = -(\Delta/2) + V$ with the square-well defined just above. By (2.7) we have

$$(f, e^{-tH} g) = \int_{\Omega_0} f^*(0)\, g[x_w(t)]\, e^{-\int_0^t V[x_w(s)]\,ds}\, Db \tag{2.9}$$

where Db is the Wiener measure (the motion starts at zero). Choosing $f(x) = g(x) = 1 \in L^\infty$, (2.9) becomes, letting $t \to \infty$

$$\lim_{t \to \infty} (f, e^{-tH} g) = \lim_{t \to \infty} \int_{\Omega_0} e^{\lambda |(s:\, |\vec{x}_w(s)| \leqslant L;\, s \leqslant t)|}\, Db \tag{2.10}$$

Recurrence property (see introduction) implies that for $d = 1, 2$, $\exp \lambda\, |(\cdot)| \to \infty$ as $t \to \infty$, so as a map from L^∞ to L^∞, $\|\exp(-tH)\|$ diverges as $t \to \infty$ no matter how small λ is. But the existence of a negative bound state is equivalent to $\|\exp(-tH)\|$ diverging as a map from L^2 to L^2, and the former divergence does not imply the latter one.

This section makes complete Kac's argument: by strengthening (2.10), we prove that $\|\exp(-tH)\|$ actually diverges from L^2 to L^2.

Our proof is based on a sequence of inequalities which contradict the fact that $H = -(\Delta/2) - \lambda I\{|\vec{x}| \leqslant L\} \geqslant 0$, or equivalently, that $\exp(-tH)$ is bounded from L^2 to L^2. Once we know that H cannot be positive, we will know that there is some (negative energy) bound state, because, by Weyl's essential spectrum theorem, the essential spectrum of H and of H_0 are the same.[7]

For $d=1$, Itô and MacKean (Ref. 6, p. 230) proved the following theorem concerning the asymptotic distribution of $\varepsilon(t, w)$ defined in (2.2)

$$\lim_{t\to\infty} P_y\left[\frac{\varepsilon(t, w)}{e(\mathbb{R})\sqrt{t}} \leqslant u\right] = \sqrt{\frac{2}{\pi}} \int_0^u e^{-t^2/2}\, dt \tag{2.11}$$

for $u \geqslant 0$, $y \in \mathbb{R}$, and $e(\mathbb{R}) = \int e(dx)$, where e is the measure used to compute $\varepsilon(t, w)$ in (2.2).

So we write

$$0 < \sqrt{\frac{2}{\pi}} \int_u^\infty e^{-t^2/2}\, dt = \lim_{t\to\infty} P_y\left[\frac{\varepsilon(t, w)}{\sqrt{t}} \geqslant ue(\mathbb{R})\right] \tag{2.12}$$

and choosing $e(dx) = I\{|x| \leqslant L\}\, dx$, $e(\mathbb{R}) = 2L$ ($\varepsilon(t, w)$ then equals the time spent in $[-L, L]$ during a time t), (2.12) is equal to

$$= \lim_{t\to\infty} P_y\left[\frac{\varepsilon(t, w)}{\sqrt{t}} \geqslant 2uL\right] \tag{2.13}$$

$$\begin{aligned} &= \lim_{t\to\infty} P_y\left[\frac{\varepsilon(t, w)}{\sqrt{t}} \geqslant 2uL \mid |x_w(t)| \leqslant at^\alpha\right] \cdot P_y[|x_w(t)| \leqslant at^\alpha] \\ &+ \lim_{t\to\infty} P_y\left[\frac{\varepsilon(t, w)}{\sqrt{t}} \geqslant 2uL \mid |x_w(t)| \geqslant at^\alpha\right] \cdot P_y[|x_w(t)| \geqslant at^\alpha] \end{aligned} \tag{2.14}$$

For the one-dimensional Wiener measure, we have

$$P_y[x_w(t) \in dx] = \frac{dx}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}$$

so that, if $\alpha > \frac{1}{2}$ and $a > 0$

$$P_y[|x_w(t)| \geqslant at^\alpha] = \sqrt{\frac{2}{\pi t}} \int_{at^\alpha}^\infty \exp\left[\frac{(x-y)^2}{2t}\right] dx \to 0 \qquad t \to \infty \tag{2.15}$$

Therefore the second term of (2.14) vanishes in the limit $t \to \infty$. One has also

$$\lim_{t\to\infty} P_y[|x_w(t)| \leqslant at^\alpha] = 1 - \lim_{t\to\infty} P_y[|x_w(t)| \geqslant at^\alpha] = 1 \tag{2.16}$$

Furthermore, since (2.12) is independent of y, we write (2.14) as

$$\frac{1}{2a}\int_{-\infty}^{\infty} dy \lim_{t\to\infty}\int_{-\infty}^{\infty} f(y)\, g(z)\, P_y\left[\frac{\varepsilon(t, w)}{\sqrt{t}} \geqslant 2uL \mid x_w(t)\in dz\right] \quad (2.17)$$

in which we set $f(y)=I\{|y|\leqslant a\}$, $g(z)=I\{|z|\leqslant at^{\alpha}\}$ and $a>0$.

$$(2.17)=\frac{1}{2a}\int_{-\infty}^{\infty} dy \lim_{t\to\infty}\int_{-\infty}^{\infty} dz\, f(y)\, g(z)\int_A Db(y, 0; z, t) \quad (2.18)$$

with $A=\{w\in\Omega_y\colon \varepsilon(t, w)\geqslant 2uL\sqrt{t}\}\subset\Omega_y$. For $w\in A$, we have the following inequality

$$1\leqslant e^{-2u\lambda L\sqrt{t}}e^{-\int_0^t V[x_w(s)]\, ds} \quad (2.19)$$

so that

$$(2.18)\leqslant\frac{1}{2a}\int_{-\infty}^{\infty} dy \lim_{t\to\infty}\int_{-\infty}^{\infty} dz\, f(y)\, g(z)$$

$$\times e^{-2u\lambda L\sqrt{t}}\int_A e^{-\int_0^t V[x_w(s)]\, ds}\, Db(y, 0; z, t) \quad (2.20)$$

$$\leqslant\frac{1}{2a}\int_{-\infty}^{\infty} dy \lim_{t\to\infty}\int_{-\infty}^{\infty} dz\, f(y)\, g(z)$$

$$\times e^{-2u\lambda L\sqrt{t}}\int_{\Omega_y} e^{-\int_0^t V[x_w(s)]\, ds}\, Db(y, 0; z, t) \quad (2.21)$$

$$=\frac{1}{2a}\int_{-\infty}^{\infty} dy \lim_{t\to\infty} e^{-2u\lambda L\sqrt{t}}\int_{-\infty}^{\infty} dz\, f(y)\, g(z)\, e^{-tH}(y, z) \quad (2.22)$$

Now suppose $H\geqslant 0$. Then $\exp(-tH)(y, z)$ is bounded for $t>0$ and since $f(y)\exp(-tH)(y, z)$ is integrable for every $t>0$, we can interchange the integral on y and the limit on t.

$$(2.18)=\frac{1}{2a}\lim_{t\to\infty} e^{-2u\lambda L\sqrt{t}}(f, e^{-tH}g) \quad (2.23)$$

Since $H\geqslant 0$, the norm of $\exp(-tH)$ as an operator on L^2 is not greater than 1 and consequently

$$(2.18)\leqslant\frac{1}{2a}\lim_{t\to\infty} e^{-2u\lambda L\sqrt{t}}\|f\|\cdot\|g\| \quad (2.24)$$

$$=\lim_{t\to\infty} e^{-2u\lambda L\sqrt{t}}t^{\alpha/2} \quad (2.25)$$

$$=0 \qquad \text{for every } u, \lambda, L \text{ strictly positive } (\alpha>\tfrac{1}{2})$$

This yields the announced contradiction: we proved that $H = H_0 - \lambda I\{|x| \leqslant L\}$ possesses a bound state, for every $\lambda > 0$ and $L > 0$.

2.3. The Two-Dimensional Case

The potential we consider is again $V(\vec{x}) = -\lambda I\{|\vec{x}| \leqslant L\}$. V is spherically symmetric, so we use the two-dimensional Bessel process. In that case, Itô and Mac Kean (Ref. 6, p. 231) proved that

$$\lim_{t \to \infty} P_l \left[\frac{\varepsilon(t, w)}{\ln t} \leqslant u\mu(\mathbb{R}_+) \right] = 1 - e^{-u} \tag{2.26}$$

for $u, l \geqslant 0$ and $\mu(\mathbb{R}_+) = \int_{\mathbb{R}_+} \mu(dr)$ (see 2.4).

The following proof mimics the preceding one. We choose $\mu(dr) = I\{r \leqslant L\}\, dr$.

$$0 < e^{-u} = \lim_{t \to \infty} P_l \left[\frac{\varepsilon(t, w)}{\ln t} \geqslant uL \right] \tag{2.27}$$

$$= \lim_{t \to \infty} P_l \left[\frac{\varepsilon(t, w)}{\ln t} \geqslant uL \mid r_w(t) \leqslant 2t^\alpha \right] \cdot P_l[r_w(t) \leqslant 2t^\alpha]$$

$$+ \lim_{t \to \infty} P_l \left[\frac{\varepsilon(t, w)}{\ln t} \geqslant uL \mid r_w(t) \geqslant 2t^\alpha \right] \cdot P_l[r_w(t) \geqslant 2t^\alpha] \tag{2.28}$$

The transition probability of a d-Bessel process reads (Ref. 6, p. 59)

$$P_l[r_w(t) \in dm] = \frac{m^{d-1}}{t} (ml)^{1-(d/2)} e^{-(l^2+m^2)/2t} I_{(d/2)-1}\left(\frac{lm}{t}\right) dm \tag{2.29}$$

which reduces in $d = 2$ to

$$P_l[r_w(t) \in dm] = (m/t)\, e^{-(l^2+m^2)/2t} I_0(lm/t)\, dm \tag{2.30}$$

where I_n is a modified Bessel function of the first kind.

Due to Markov properties of the Bessel process

$$\lim_{t \to \infty} P_l[r_w(t) \geqslant 2t^\alpha]$$

does not depend on the starting point l. Hence if $\alpha > \frac{1}{2}$

$$\lim_{t \to \infty} P_0[r_w(t) \geqslant 2t^\alpha] = \lim_{t \to \infty} \frac{1}{t} \int_{2t^\alpha}^{\infty} m e^{-m^2/2t}\, dm = 0$$

Therefore

$$0 < e^{-u} = \lim_{t \to \infty} P_l \left[\frac{\varepsilon(t, w)}{\ln t} \geqslant uL \mid r_w(t) \leqslant 2t^{\alpha} \right] \tag{2.31}$$

$$= \frac{1}{a} \int_0^{\infty} dl \lim_{t \to \infty} \int_0^{\infty} dm\, f(l)\, g(m) \int_B Dv(l, 0; m, t) \tag{2.32}$$

where $f(l) = I\{l \leqslant a\}$, $g(m) = I\{m \leqslant 2t^{\alpha}\}$, $a > 0$ and $B = \{w \in \Omega_l : \varepsilon(t, w) \geqslant uL \ln t\} \subset \Omega_l$

$$(2.32) \leqslant \frac{1}{a} \int_0^{\infty} dl \lim_{t \to \infty} \int_0^{\infty} dm\, f(l)\, g(m)$$

$$\times e^{-u\lambda L \ln t} \int_{\Omega_l} e^{-\int_0^t V[r_w(s)]\, ds}\, Dv(l, 0; m, t) \tag{2.33}$$

$$= \frac{1}{a} \int_0^{\infty} dl \lim_{t \to \infty} \int_0^{\infty} dm\, f(l)\, g(m)\, t^{-u\lambda L} e^{-tH_r}(l, m) \tag{2.34}$$

with $H_r = \frac{1}{2}[d_r^2 + (1/r)\, d_r] + V(r)$.

Suppose that $H \geqslant 0$ and therefore that $H_r \geqslant 0$.

$$(2.34) = \frac{1}{a} \lim_{t \to \infty} t^{-u\lambda L} \, \|f\|_{L^2(\mathbb{R}_+, rdr)} \cdot \|g\|_{L^2(\mathbb{R}_+, rdr)} \tag{2.35}$$

$$= \lim_{t \to \infty} t^{\alpha - u\lambda L} \tag{2.36}$$

$$= 0 \qquad \text{if} \quad u\lambda L > \alpha > \frac{1}{2}$$

The condition $u\lambda L > \alpha$ is not a restriction, because we can satisfy it by choosing the value of u. Therefore, the same conclusion as the one-dimensional case holds: $H = H_0 - \lambda I\{|\vec{x}| \leqslant L\}$ has at least one bound state, for any $\lambda, L > 0$.

3. SOME REMARKS ABOUT ONE- AND TWO-DIMENSIONAL BROWNIAN LOCAL TIME

In this section, we make some remarks concerning a one- and two-dimensional square-well potential and related Brownian quantities. As a consequence of the Feynman–Kac formula (2.5), we can express the lowest eigenvalue of $H = -(\Delta/2) + V$ as

$$\inf(\operatorname{spec} H) = -\lim_{t \to \infty} \frac{1}{t} \ln E_x\left(e^{-\int_0^t V[x_w(s)]\, ds}\right) \tag{3.1}$$

Letting $H = -(\Delta/2) - \lambda I\{|\vec{x}| \leqslant L\}$,

$$\inf(\operatorname{spec} H) = -\lim_{t \to \infty} \frac{1}{t} \ln E_x(e^{\lambda \varepsilon(t, w)}) \tag{3.2}$$

where $\varepsilon(t, w)$ is really the time fraction spent in the support S of the potential V. Therefore, the ground state energy gives the asymptotic behavior of the generating function of $\varepsilon(t)$, and inversely.

We see also that a necessary and sufficient condition to get a bound state is the exponential divergence of the generating function $E_x(e^{\lambda\varepsilon(t)})$. This divergence results from two conflicting effects: the paths that spend a long time in S and thus make $\exp[\lambda\varepsilon(t)]$ large tend to have a small probability with respect to the Brownian measure. Indeed we know that the average of $\varepsilon(t)$ is proportional to $\sqrt{t}$ $(d=1)$ or $\ln t$ $(d=2)$ (see below). The existence of the bound state is due to paths that spend a much longer time in S than these averages, namely a time $\sim ct$.

These contribute to the "partition function" $E_x(\exp\{-\int_0^t V[x_w(s)]\,ds\})$ a factor $\exp(\lambda\, ct)$. The question is then: What is the probability of these paths according to the Brownian measure? If it were as small as $\exp(-c't)$ (which happens when the motion is transient, i.e., in $d > 2$), then this would control the divergence of $\exp(\lambda ct)$ for λ small, and so we would not have a bound state. One way to prove the presence of a bound state would therefore be to show directly that the weight of the paths that spend in S a time proportional to t is not too small, i.e., is larger than $\exp[-f(t)]$ with $f(t)/t \to 0$ as $t \to \infty$.

We have not quite followed this idea: rather, we used estimates on deviations around the $\sqrt{t}$ or $\ln t$ average behavior and this turned out to be sufficient to prove the exponential divergence of $E_x(e^{\lambda\varepsilon(t)})$. This implied that H is not positive and, using Weyl's theorem, that H possesses a bound state.

Our argument can be regarded as an entropy–energy argument: we show, for a well-chosen set of paths, that the divergence of the potential term wins over the damping coming from the kinetic term (the Brownian measure). One might naively identify the potential with an energy term and the Brownian measure as an entropy term (since it is a measure on paths, i.e., on configurations). However, in statistical mechanics, we know that the entropy dominates the energy in low dimensions and what we just said sounds like the opposite.

Recall first that the Brownian measure (free measure in quantum mechanics) can be expressed as an integration over all paths of $\exp(\int_0^t (\dot{x}^2/2)(s)\,ds)$ [4]. If we discretize the functional integral in (3.1), we see that the Brownian measure becomes $\exp[\frac{1}{2}\sum_i (x_{i+1} - x_i)^2]$ and therefore couples the variables x_is for different is (interaction with the nearest neighbors). Hence the Brownian measure corresponds to the energy from a statistical viewpoint. The potential term becomes the "single-spin" distribution $\exp[-\sum_i V(x_i)]$ and is analogous to an entropy term.

Another analogy between the quantum and statistical mechanical situation is the following one. Let the coupling constant correspond to temperature T. For λ (resp., T) large, we have a bound state (resp., a disordered phase). The relation between high-temperature expansions and perturbation theory in quantum mechanics was stressed by Faris.[3]

For λ small and $d>2$ (resp., T small) we have a continuous spectrum for H (resp., an ordered phase), while a bound state persists for all $\lambda>0$ if $d\leqslant 2$ (resp., a disordered phase for statistical systems with a continuous internal symmetry).

It is a standard exercise to solve the Schrödinger equation for $H=-(\Delta/2)-\lambda I\{|\vec{x}|\leqslant L\}$.

The energy levels are determined by the solutions of transcendental equations ($|E|$ is the absolute value of the energy E)

$$\begin{aligned} d&=1 \qquad & |E| &= \lambda \sin^2(\alpha L) \\ d&=2 \qquad & \alpha J_1(\alpha L)\, K_0(\beta L) &= \beta J_0(\alpha L)\, K_1(\beta L) \end{aligned} \tag{3.3}$$

in which $\alpha=\sqrt{2(\lambda-|E|)}$, $\beta=\sqrt{2\,|E|}$, J_i, and K_i are Bessel functions. For small λ and L, we can perturbatively solve these equations. The first one yields, for the ground state energy

$$E_0=-2\lambda^2L^2+\frac{16}{3}\lambda^3L^4-\frac{736}{45}\lambda^4L^6+\cdots \tag{3.4}$$

(By implicit functions theorem, 3.4 is the unique solution in the neighborhood of $\lambda=L=0$.)

The second implicit equation gives, at first order in λ, L

$$E_0\simeq -\frac{2}{L^2}\exp\left(-\frac{2}{\lambda^2L^2}-2\gamma\right) \tag{3.5}$$

in which $\gamma=0.57721...$ is the Euler constant.

We remark that the ground-state energy is much closer to zero in $d=2$ than in $d=1$, since if, e.g., $\lambda=L=0.01$, one finds that

$d=1$	$E_0\sim -2.10^{-8}$	atomic units
$d=2$	$E_0\sim -2.10^{-868586}$	atomic units!!

The energy values (3.4) and (3.5) lead to the generating functions

$$d=1 \qquad E(e^{\lambda\varepsilon(t)})=e^{-2\lambda^2L^2t} \qquad \text{for} \quad t,\frac{1}{\lambda},\frac{1}{L}\gg 1$$

$$d=2 \qquad E(e^{\lambda\varepsilon(t)})=\exp\left[-\frac{2t}{L^2}\exp\left(-\frac{2}{\lambda^2L^2}-2\gamma\right)\right]$$

$$\text{for} \quad t,\frac{1}{\lambda},\frac{1}{L}\gg 1 \tag{3.6}$$

From (3.6), we can compute the asymptotic distribution of the time spent in $[-L, L]$, for $d=1$ only

$$d=1 \quad P[\varepsilon(t)\in ds]=\frac{ds}{\sqrt{2\pi L^2t}}\exp\left(-\frac{s^2}{8L^2t}\right) \quad s,t,\frac{1}{\lambda},\frac{1}{L}\gg 1 \tag{3.7}$$

By (2.5) and (2.6), we can compute the first moments of $\varepsilon(t)$. We take for $d=1$, $f(x)=1$, $k(x)=\lambda I\{|x|\leqslant L\}$ and $\theta=\frac{1}{2}d_x^2$, and for $d=2$, $f(r)=1$, $k(r)=\lambda\{r\leqslant L\}$ and $\theta=\frac{1}{2}(d_r^2+(1/r)\,d_r)$.

It is difficult to obtain explicit formulas for these moments which are infinite series, but we can compute the main terms, as $t\gg 1$. These are

$$d=1 \qquad E_0[\varepsilon^{2n}(t)]=\frac{(2n)!}{n!}(2tL^2)^n+O(t^{n-1/2})$$

$$E_0[\varepsilon^{2n+1}(t)]=\frac{n!}{\sqrt{\pi}}(8L^2t)^{n+1/2}+O(t^n) \tag{3.8}$$

$$d=2 \qquad E_0[\varepsilon(t)]=\frac{L^2}{2}\ln\frac{t}{L^2}+O(1)$$

$$E_0[\varepsilon^2(t)]=2L^4\ln^2\frac{t}{L^2}+O(\ln t) \tag{3.9}$$

$$E_0[\varepsilon^3(t)]=\frac{183}{4}L^4t\ln\frac{t}{L^2}+O(t)$$

We observe that the asymptotic generating function (3.6) in $d=1$ yields the correct asymptotic moments of $\varepsilon(t)$ (of even order only, since 3.6 is even in λ) when expanded in powers of λ, despite a nonobvious exchange of limits (in t and in λ). In $d=2$ however, the asymptotic generating function does not allow us to say anything about asymptotic moments.

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